Loan Guarantees
Part II - The Revised BSOPM - Model Basics

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In Part II to the series on Loan Guarantee we will build a continuous time model to value a loan guarantee with and without a cap. The valuation equations will look a lot like the Black-Scholes put option pricing model but with some revisions.

Equations for Enterprise Value

In Part I we defined the variable A_t to be enterprise value at time t, the variable κ to be the cost of capital, the variable ϕ to be dividend yield, the variable σ to be return volatility, and the variable W_t to be the value of a Brownian motion at time t. The stochastic differential equation that defines how enterprise value changes over time is...

$$\delta A_t = \left(\kappa - \phi\right) A_t \, \delta t + \sigma \, A_t \, \delta W_t \text{ ...where... } \delta W_t \sim N\left[0, \delta t\right] \tag{1}$$

The solution to Equation (1) above is the equation for enterprise value at time t, which is...

$$A_t = A_0 \operatorname{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\sigma^2\right) t + \sigma \sqrt{t}\, Z\right\} \text{ ...where... } Z \sim N\left[0, 1\right] \tag{2}$$

We will define the function $P(Z)$ to be the probability density function of the normal distribution under the actual probability Measure P where the variable m is the mean of the distribution and the variable v is the variance. The equation for the probability density function applicable to asset price Equation (2) above is... [1]

$$P(Z) = \sqrt{\frac{1}{2\,\pi\,v}} \operatorname{Exp}\left\{-\frac{1}{2\,v}\left(Z - m\right)^2\right\} \text{ ...where... } m = 0 \text{ ...and... } v = 1 \tag{3}$$

In Part I we defined the variable α to be the risk-free rate. We will define the variable n to be the mean of the distribution of the random variable Z in Equations (2) and (3) above under the risk-neutral Measure Q. Under the risk-neutral measure all assets earn the risk-free rate. The equation for the mean of the distribution of Z under the risk-neutral Measure Q is... [2]

$$n = \frac{(\alpha - \kappa)\, t}{\sigma \sqrt{t}} \tag{4}$$

We will the function $X(Z)$ to be the Girsanov multiplier, which we will use to move the mean of the distribution of the normally-distributed random variable Z from m, which is the mean of the actual probability Measure P, to n, which is the mean of the risk-neutral probability Measure Q. Using Equations (2), (3) and (4) above the equation for the Girsanov multipler is... [2]

$$X(Z) = \operatorname{Exp}\left\{\frac{n - m}{v}\, Z - \frac{n^2 - m^2}{2\,v}\right\} \tag{5}$$

We will define the function $Q(Z)$ to be the probability density function of the normal distribution under the risk-neutral probability Measure Q. Using Appendix Equation (25) below the equation for the probability density function is...

$$Q(Z) = P(Z)\, X(Z) = \sqrt{\frac{1}{2\,\pi\,v}} \operatorname{Exp}\left\{-\frac{1}{2\,v}\left(Z - n\right)^2\right\} \tag{6}$$

Equations for Guarantee Value - Uncapped

We will define the variable G_0 to be the value of a loan guarantee at time zero, the variable Γ to be the liquidation value of each dollar of going-concern value, and the variable D_t to be the debt payoff amount at time t. Using Equations (2) and (6) above the equation for the no-arbitrage value of a loan guarantee is...

$$G_0 = \int\limits_{-\infty}^{\infty} Q(Z)\,\mathrm{Max}\Big(D_t - \Gamma A_t, 0\Big)\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \;\;\text{...where...}\;\; Z \sim N\Big[n, v\Big] \tag{7}$$

We will define the variable a to be the value of the random variable Z where enterprise value equals the debt payoff amount (i.e. the default point). Using Equation (2) above the equation for the default point as...

$$\text{if...}\;\; A_0\,\mathrm{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right)t + \sigma\sqrt{t}\,Z\right\} = D_t \;\;\text{...then...}\;\; a = \left[\ln\left(\frac{D_t}{A_0}\right) - \left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} \tag{8}$$

Using the definitions in Equation (8) above we can rewrite Equation (7) above as...

$$G_0 = \int\limits_{-\infty}^{a} Q(Z)\Big(D_t - \Gamma A_t\Big)\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \;\;\text{...where...}\;\; Z \sim N\Big[n, v\Big] \tag{9}$$

Using Equations (6) and (9) above we will make the following integral definitions...

$$I_1 = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z - n\Big)^2\right\} D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z$$

$$I_2 = \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\Big(Z - n\Big)^2\right\} A_0\,\mathrm{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right)t + \sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \tag{10}$$

Using the integral definitions in Equation (10) above we can rewrite Equation (9) above as...

$$G_0 = I_1 - \Gamma\,I_2 \tag{11}$$

Using Appendix Equations (30) and (35) below we can rewrite Equation (11) above as...

$$G_0 = D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,CND\Big[a - n\Big] - \Gamma A_0\,\mathrm{Exp}\Big\{-\phi\,t\Big\}\,CND\Big[a - (n + \sigma\sqrt{t})\Big] \tag{12}$$

Using Equations (4) and (8) above we will define the variable d_1 as follows...

$$\begin{aligned} d_1 &= a - n \\ &= \left[\ln\left(\frac{D_t}{A_0}\right) - \left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} - (\alpha - \kappa)\,t\Big/\sigma\sqrt{t} \\ &= \left[\ln\left(\frac{D_t}{A_0}\right) - \left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} \end{aligned} \tag{13}$$

Using Equations (4), (8) and (13) above we will define the variable d_2 as follows...

$$d_2 = a - (n + \sigma\sqrt{t}) = d_1 - \sigma\sqrt{t} \tag{14}$$

Using Equations (13) and (14) above we can rewrite Equation (12) above as...

$$G_0 = D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,CND\Big[d_1\Big] - \Gamma A_0\,\mathrm{Exp}\Big\{-\phi\,t\Big\}\,CND\Big[d_2\Big] \tag{15}$$

Equations for Guarantee Value - Capped

If the Guarantor's payment under the terms of the guarantee is capped at CAP then Equation (7) above becomes...

$$G_0 = \int_{-\infty}^{\infty} Q(\theta_t)\,\mathrm{Min}\Big(\mathrm{Max}\Big(D_t - \Gamma A_t, 0\Big), CAP\Big)\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \tag{16}$$

We will define the variable b to be the value of the random variable Z where debt payoff amount minus enterprise value equals the capped amount (i.e. the cap point). Using Equations (2) and (8) above the equation for the cap point as...

$$\text{if... } A_t - D_t = CAP \text{ ...then... } b = \mathrm{Min}\Bigg(\Bigg[\ln\Bigg(\frac{D_t - CAP}{\Gamma A_0}\Bigg) - \Big(\kappa - \phi - \frac{1}{2}\,\sigma^2\Big)t\Bigg]\Big/\sigma\sqrt{t},\, a\Bigg) \tag{17}$$

Using the definitions in Equation (17) above we can rewrite Equation (16) above as...

$$G_0 = \int_{b}^{a} Q(Z)\Big(D_T - \Gamma A_t\Big)\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z + \int_{-\infty}^{b} Q(Z)\,CAP\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \tag{18}$$

Using Equations (6) and (18) above we will make the following integral definitions...

$$I_3 = \int_{b}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\Big\{-\frac{1}{2}\Big(Z-n\Big)^2\Big\}D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z$$

$$I_4 = \int_{b}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\Big\{-\frac{1}{2}\Big(Z-n\Big)^2\Big\}A_0\,\mathrm{Exp}\Big\{\Big(\kappa-\phi-\frac{1}{2}\,\sigma^2\Big)t+\sigma\sqrt{t}\,Z\Big\}\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z$$

$$I_5 = \int_{-\infty}^{b}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\Big\{-\frac{1}{2}\Big(Z-n\Big)^2\Big\}CAP\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,\delta Z \tag{19}$$

Using the integral definitions in Equation (19) above we can rewrite Equation (18) above as...

$$G_0 = I_3 - \Gamma\,I_4 + I_5 \tag{20}$$

Using Appendix Equations (36), (37) and (38) below we can rewrite Equation (20) above as...

$$\begin{aligned} G_0 = {} & D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\Big(CND\Big[a-n\Big] - CND\Big[b-n\Big]\Big) - \\ & \Gamma A_0\,\mathrm{Exp}\Big\{-\phi\,t\Big\}\Big(CND\Big[a-(n+\sigma\sqrt{t})\Big] - CND\Big[b-(n+\sigma\sqrt{t})\Big]\Big) + \\ & CAP\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,CND\Big[b-n\Big] \end{aligned} \tag{21}$$

Using Equations (4) and (17) above we will define the variable d_3 as follows...

$$\begin{aligned} d_3 &= b - n \\ &= \Bigg[\ln\Bigg(\frac{D_t - CAP}{\Gamma A_0}\Bigg) - \Big(\kappa-\phi-\frac{1}{2}\,\sigma^2\Big)t\Bigg]\Big/\sigma\sqrt{t} - (\alpha-\kappa)\,t\Big/\sigma\sqrt{t} \\ &= \Bigg[\ln\Bigg(\frac{D_t - CAP}{\Gamma A_0}\Bigg) - \Big(\alpha-\phi-\frac{1}{2}\,\sigma^2\Big)t\Bigg]\Big/\sigma\sqrt{t} \end{aligned} \tag{22}$$

Using Equations (4), (17) and (22) above we will define the variable d_4 as follows...

$$d_4 = b - (n + \sigma\sqrt{t}) = d_3 - \sigma\sqrt{t} \tag{23}$$

Using Equations (13), (14), (22) and (23) above we can rewrite Equation (21) above as...

$$\begin{aligned} G_0 = {} & D_t\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\Big(CND\Big[d_1\Big] - CND\Big[d_3\Big]\Big) - \Gamma A_0\,\mathrm{Exp}\Big\{-\phi\,t\Big\}\Big(CND\Big[d_2\Big] - CND\Big[d_4\Big]\Big) + \\ & CAP\,\mathrm{Exp}\Big\{-\alpha\,t\Big\}\,CND\Big[d_3\Big] \end{aligned} \tag{24}$$

References

[1] Gary Schurman, *The Calculus of the Normal Distribution,* October, 2010.

[2] Gary Schurman, *The Girsanov Multiplier,* May, 2017.

Note that the function $CND(Z)$ in the equations above is the cumulative normal distribution function for a normally-distributed random variable Z with mean zero and variance one. The Excel equivalent function is $NORMSDIST(Z)$.

Appendix

A. Using Equations (3) and (5) above the solution to Equation (6) above is...

$$
\begin{aligned}
Q(\theta_t) &= P(\theta_t)\,X(\theta_t) \\
&= \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\left(\theta_t-m\right)^2\right\}\text{Exp}\left\{\frac{n-m}{v}\,\theta_t-\frac{n^2-m^2}{2\,v}\right\} \\
&= \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\left(\theta_t^2-2\,m\,\theta_t+m^2\right)\right\}\text{Exp}\left\{-\frac{1}{2\,v}\left(2\,m\,\theta_t-2\,n\,\theta_t+n^2-m^2\right)\right\} \\
&= \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\left(\theta_t^2-2\,n\,\theta_t+n^2\right)\right\} \\
&= \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\left(\theta_t-n\right)^2\right\}
\end{aligned}
\tag{25}
$$

B. Using Equation (10) above the solution to I_1 in Equation (11) above is...

$$
\begin{aligned}
I_2 &= \int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\pi}}\,\text{Exp}\left\{-\frac{1}{2}\left(Z-n\right)^2\right\}A_0\,\text{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,Z\right\}\text{Exp}\left\{-\alpha\,t\right\}\delta Z \\
&= A_0\,\text{Exp}\left\{\left(\kappa-\alpha-\phi-\frac{1}{2}\sigma^2\right)t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\pi}}\,\text{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,n\,Z+n^2\right)\right\}\text{Exp}\left\{\sigma\sqrt{t}\,Z\right\}\delta Z \\
&= A_0\,\text{Exp}\left\{\left(\kappa-\alpha-\phi-\frac{1}{2}\sigma^2\right)t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\pi}}\,\text{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,(n+\sigma\sqrt{t})\,Z+n^2\right)\right\}\delta Z
\end{aligned}
\tag{26}
$$

We will make the following definition...

$$
\theta = Z-(n+\sigma\sqrt{t})\ \text{ ...such that... }\ \theta^2 = Z^2-2\,(n+\sigma\sqrt{t})\,Z+n^2+2\,n\,\sigma\sqrt{t}+\sigma^2 t \tag{27}
$$

Using the definitions in Equation (27) above we can rewrite Equation (26) above as...

$$
\begin{aligned}
I_2 &= A_0\,\text{Exp}\left\{\left(\kappa-\alpha-\phi-\frac{1}{2}\sigma^2\right)t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\pi}}\,\text{Exp}\left\{-\frac{1}{2}\left(\theta^2-2\,n\,\sigma\sqrt{t}-\sigma^2 t\right)\right\}\delta Z \\
&= A_0\,\text{Exp}\left\{\left(\kappa-\alpha-\phi-\frac{1}{2}\sigma^2\right)t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\text{Exp}\left\{\frac{1}{2}\,\sigma^2 t+n\,\sigma\sqrt{t}\right\}\delta Z \\
&= A_0\,\text{Exp}\left\{\left(\kappa-\alpha-\phi-\frac{1}{2}\sigma^2\right)t\right\}\text{Exp}\left\{\frac{1}{2}\,\sigma^2 t+\alpha\,t-\kappa\,t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z \\
&= A_0\,\text{Exp}\left\{-\phi\,t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z
\end{aligned}
\tag{28}
$$

The equation for the derivative of Equation (27) above is...

$$\frac{\delta\theta}{\delta Z} = 1 \text{ ...such that... } \delta\theta = \delta Z \tag{29}$$

Using Equation (29) above we can rewrite Equation (28) above as...

$$\begin{aligned} I_2 &= A_0 \operatorname{Exp}\left\{ -\phi\, t \right\} \int\limits_{-\infty-(n+\sigma\sqrt{t})}^{a-(n+\sigma\sqrt{t})} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta \\ &= A_0 \operatorname{Exp}\left\{ -\phi\, t \right\} \int\limits_{-\infty}^{a-(n+\sigma\sqrt{t})} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta \\ &= A_0 \operatorname{Exp}\left\{ -\phi\, t \right\} CND\left[a - (n + \sigma\sqrt{t}) \right] \end{aligned} \tag{30}$$

C. Using Equation (10) above the solution to I_2 in Equation (11) above is...

$$\begin{aligned} I_1 &= \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\left(Z - n \right)^2 \right\} D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \delta Z \\ &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\left(Z^2 - 2\,n\,Z + n^2 \right) \right\} \delta Z \end{aligned} \tag{31}$$

We will make the following definition...

$$\theta = Z - n \text{ ...such that... } \theta^2 = Z^2 - 2\,n\,Z + n^2 \tag{32}$$

Using the definitions in Equation (32) above we can rewrite Equation (31) above as...

$$I_1 = D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta Z \tag{33}$$

The equation for the derivative of Equation (32) above is...

$$\frac{\delta\theta}{\delta Z} = 1 \text{ ...such that... } \delta\theta = \delta Z \tag{34}$$

Using Equation (34) above we can rewrite Equation (33) above as...

$$\begin{aligned} I_1 &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \int\limits_{-\infty-n}^{a-n} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta \\ &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \int\limits_{-\infty}^{a-n} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta \\ &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} CND\left[a - n \right] \end{aligned} \tag{35}$$

D. Using Equation (35) above the solution to I_3 in Equation (20) above is...

$$\begin{aligned} I_3 &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \int\limits_{b-n}^{a-n} \sqrt{\frac{1}{2\,\pi}} \operatorname{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta \\ &= D_t \operatorname{Exp}\left\{ -\alpha\, t \right\} \left(CND\left[a - n \right] - CND\left[b - n \right] \right) \end{aligned} \tag{36}$$

E. Using Equation (30) above the solution to I_4 in Equation (20) above is...

$$I_4 = A_0 \,\mathrm{Exp}\left\{ -\phi\, t \right\} \int\limits_{b-(n+\sigma\sqrt{t})}^{a-(n+\sigma\sqrt{t})} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta$$

$$= A_0 \,\mathrm{Exp}\left\{ -\phi\, t \right\} \left(CND\left[a - (n + \sigma\sqrt{t}) \right] - CND\left[b - (n + \sigma\sqrt{t}) \right] \right) \tag{37}$$

F. Using Equation (35) above the solution to I_5 in Equation (20) above is...

$$I_5 = CAP \,\mathrm{Exp}\left\{ -\alpha\, t \right\} \int\limits_{-\infty}^{b-n} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{ -\frac{1}{2}\,\theta^2 \right\} \delta\theta$$

$$= CAP \,\mathrm{Exp}\left\{ -\alpha\, t \right\} CND\left[b - n \right] \tag{38}$$